NUMERICAL SOLUTION OF A FAST DIFFUSION EQUATION

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ABSTRACT. In this paper, the authors consider the first boundary value problem for the nonlinear reaction diffusion equation: $u_t - \Delta u^m = \alpha u^{p_1}$ in Ω, a smooth bounded domain in $\mathbb{R}^d (d \geq 1)$ with the zero lateral boundary condition and with a positive initial condition, $m \in]0, 1[$ (fast diffusion problem), $\alpha \geq 0$ and $p_1 \geq m$. Sufficient conditions on the initial data are obtained for the solution to vanish or become infinite in a finite time. A scheme for the discretization in time of this problem is proposed. The numerical scheme preserves the essential properties of the initial problem; namely existence of an extinction or a blow-up time, for which estimates have been obtained. The convergence of the method is also proved.

1. INTRODUCTION

In this paper, a numerical scheme is proposed to solve the reaction diffusion problem: find a nonnegative function u defined on Ω, a smooth domain $\mathbb{R}^d (d \geq 1)$ and such that

$$(1.1) \quad \begin{aligned} u_t - \Delta u^m &= \alpha u^{p_1} & x \in \Omega, t > 0 \\ u(t, x) &= 0 & x \in \partial\Omega, \ t \geq 0 \\ u(0, x) &= u_0(x) > 0 & x \in \Omega, \end{aligned}$$

where $m \in]0, 1[$ (fast diffusion problem), $\alpha > 0$ and $p_1 \geq m$.

This problem and analogous problems have been studied from a theoretical point of view by several authors: Aronson-Crandall-Peletier [2], Berryman-Holland [3], Friedman-Lacey [4], Friedman-McLeod [5], [6], Levine-Sacks [13], Sabinina [15], and Sacks [16], [17], [18]. M.-N. Le Roux has proposed a numerical method in [8] and [9] to compute the solution of a similar problem (1.1) with $m > 1$ (slow diffusion problem).

In the case $\alpha = 0$, there exists an extinction time T^* such that the problem (1.1) has a unique classical solution, positive on $\Omega \times [0, T^*[$ and null for $t \geq T^*$ (see [3], [15]). Concerning this last case, a semidiscretization in time is proposed in [10] by M.-N. Le Roux, for which the numerical solution has the same properties as the exact solution, so it allows the calculation of a numerical extinction time.

In the case $\alpha > 0$, according to the values of p_1, the solution of (1.1) may vanish or blow up in a finite time:

- For $p_1 \in [m, 1]$, the solution of (1.1) cannot blow up, but it may vanish in some finite time;

- For $p_1 > 1$, it is possible for the solution to vanish or blow up in some finite time according to the initial data.

Here, we study the semidiscretization in time of the problem (1.1): we propose a scheme whose solution has the same properties as the solution of the theoretical problem, in particular extinction or blow up in a finite time. A complete discretization of (1.1) using a P_1-finite element method has been studied in [14]; the results obtained are the same as for the semidiscretization in time and the proofs are analogous, so, we shall not develop this point. Further, numerical results concerning this problem or similar problems may be found in [12] or [14].

An outline of this paper follows.

In §2, we recall some theoretical results and set up sufficient conditions on the initial data for the solution to vanish or blow up in a finite time.

In §3, we define a numerical scheme for the semidiscretization in time of (1.1) and we prove the existence of the numerical solution. An iterative method to solve the nonlinear equation obtained at each time step is proposed and its convergence is proved.

In §4, we study the behavior of the numerical solution; it has similar properties as the exact solution.

In §5, we prove the convergence of the numerical method.

2. ASYMPTOTIC BEHAVIOR OF THE SOLUTION

By using the variable $v = u^m$, it is more convenient to work with the transformed equation

$$(2.1) \qquad \begin{aligned} pv^{p-1}v_t + Av &= \alpha v^r, & t > 0, \\ v(0) &= v_0 = u_0^m > 0, \end{aligned}$$

where $p = \frac{1}{m}$, $r = \frac{p_1}{m}$; so $p > 1$ and $r \geq 1$. A denotes the operator $-\Delta$ of domain $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

This problem has a unique solution at least on a bounded interval ([16]).

We suppose that p satisfies the following hypothesis:

$$(2.2) \qquad p > 1 \text{ if } d \leq 2; \ 1 < p < \frac{d+2}{d-2} \text{ if } d > 2, \qquad (H_1)$$

which assures continuous and compact embedding of $H_0^1(\Omega)$ in $L^{p+1}(\Omega)$. Then, we denote

$$(2.3) \qquad C(\Omega) = \mathrm{Inf}_{\varphi \in H_0^1(\Omega), \varphi \neq 0} \frac{1}{\|\varphi\|_{p+1}^2} \int_\Omega |\nabla \varphi|^2 dx.$$

We suppose also that the initial condition is in $H_0^1(\Omega) \cap C^\epsilon(\overline{\Omega})$, $\epsilon > 0$.

For $s \geq 1$, we denote by $\|.\|_s$ the natural norm in $L^s(\Omega)$ and by $\|.\|_\infty$ the one of $L^\infty(\Omega)$.

Lemma 2.1. If $r \leq p$, the solution v of (2.1) exists for all $t \geq 0$ and satisfies

$$(2.4) \qquad \|v(t)\|_\infty^{p-r} \leq \|v_0\|_\infty^{p-r} + \alpha \frac{p-r}{p}t, \ \text{if } r < p,$$

$$(2.5) \qquad \|v(t)\|_\infty \leq \|v_0\|_\infty \exp(\frac{\alpha}{p}t), \ \text{if } r = p.$$

If $r > p$, the solution exists at least on the interval $[0, T_1]$, where

(2.6)
$$T_1 = \frac{1}{\alpha} \frac{p}{r - p} \frac{1}{\|v_0\|_\infty^{r-p}}$$

and satisfies

(2.7)
$$\|v(t)\|_\infty^{r-p} \le \|v_0\|_\infty^{r-p} \frac{T_1}{T_1 - t} .$$

Proof. In the same way as Sacks in [17], we obtain that the solution of (2.1) is bounded by the solution w of the ordinary differential equation

(2.8)
$$\begin{aligned} pw^{p-1}(t)w'(t) &= \alpha w^r(t), \quad t > 0, \\ w(0) &= \|v_0\|_\infty. \end{aligned}$$

The solution w is written $w(t) = \|v_0\|_\infty (1 + \alpha \frac{p-r}{p} \|v_0\|_\infty^{r-p} t)^{\frac{1}{p-r}}$ if $r \ne p$ and $w(t) = \|v_0\|_\infty \exp(\frac{\alpha}{p} t)$ if $r = p$.

So, it is defined for all $t \ge 0$ if $r \le p$ and for $t < T_1$ if $r > p$ and we deduce the estimates (2.6) and (2.7).

We introduce the Lyapunov functional J^* defined by

(2.9)
$$J^*(\varphi) = \frac{1}{2} \int_\Omega |\nabla \varphi|^2 dx - \frac{\alpha}{r + 1} \int_\Omega \varphi^{r+1} dx \qquad \forall \varphi \in H_0^1(\Omega) \cap L^{r+1}(\Omega).$$

Lemma 2.2. *The mapping $t \longmapsto J^*(v(t))$ is decreasing.*

Proof. By multiplying the first equation of (2.1) by v_t and integrating on Ω, we obtain

(2.10)
$$p \int_\Omega v^{p-1}(v_t)^2 dx + \int_\Omega \nabla v \nabla v_t dx = \alpha \int_\Omega v^r v_t dx;$$

this equality may also be written as $\frac{d}{dt} J^*(v(t)) = -p \int_\Omega v^{p-1}(v_t)^2 dx$, so the derivative in time of $J^*(v)$ is negative, which proves the result.

Lemma 2.3. *For $t \ge 0$, we have the inequality*

(2.11)
$$\|v_0\|_{p+1}^{p+1} - 2\frac{p + 1}{p} J^*(v_0)t \le \|v(t)\|_{p+1}^{p+1}.$$

Proof. By multiplying the first equation of (2.1) by v and integrating on Ω, we obtain

(2.12)
$$p \int_\Omega v^p v_t dx + \int_\Omega |\nabla v|^2 dx = \alpha \int_\Omega v^{r+1} dx,$$

that is

(2.13)
$$\frac{p}{p + 1} \frac{d}{dt} \|v(t)\|_{p+1}^{p+1} + 2J^*(v(t)) = \alpha \frac{r - 1}{r + 1} \int_\Omega v^{r+1} dx.$$

As the mapping $t \longmapsto J^*(v(t))$ is decreasing, we get $\frac{p}{p+1} \frac{d}{dt} \|v(t)\|_{p+1}^{p+1} + 2J^*(v_0) \ge 0$; we deduce immediately (2.11).

Now we show that if $J^*(v_0) \le 0$ the solution of (2.1) tends to $+\infty$ in the case $r \le p$ or blows up in a finite time in the case $r > p$.

Theorem 2.4. *If $J^*(v_0) \leq 0$ and $r \leq p$, then $\lim_{t \to +\infty} \|v(t)\|_{p+1} = +\infty$, except in the case $r = 1$ and $J^*(v_0) = 0$.*

If $J^*(v_0) \leq 0$ and $r > p$, the solution of (2.1) blows up in a finite time T_b such that

$$(2.14) \qquad\qquad T_1 \leq T_b \leq T_2$$

with

$$(2.15) \qquad\qquad T_2 = \frac{1}{\alpha} \frac{p}{r-p} \frac{r+1}{r-1} \frac{1}{C_2(\Omega) \|v_0\|_{p+1}^{r-p}},$$

where $C_2(\Omega)$ is a positive constant depending only on Ω.

Moreover, for $t \in [0, T_b[$, we have the inequality

$$(2.16) \qquad\qquad \|v_0\|_{p+1} \left[\frac{T_2}{T_2 - t} \right]^{\frac{1}{r-p}} \leq \|v(t)\|_{p+1}.$$

Proof. Let us prove the result first in the case $r < p$. The relation (2.12) may also be written as

$$(2.17)$$

$$\frac{p}{p+1} \frac{d}{dt} \|v(t)\|_{p+1}^{p+1} = -(r+1)J^*(v(t)) + \frac{r-1}{2} \int_\Omega |\nabla v(t)|^2 dx$$

as the mapping $t \longmapsto J^*(v(t))$ is decreasing, and by using the positive constant defined in (2.3), we get

$$\frac{p}{p+1} \frac{d}{dt} \|v(t)\|_{p+1}^{p+1} + (r+1)J^*(v_0) \geq \frac{r-1}{2} C(\Omega) \|v(t)\|_{p+1}^2.$$

If $J^*(v_0) \leq 0$, this inequality leads to $\|v(t)\|_{p+1}^{p-1} \geq \|v_0\|_{p+1}^{p-1} + \frac{(p-1)(r-1)}{2p} C(\Omega)t$, then $\lim_{t \to +\infty} \|v(t)\|_{p+1} = +\infty$.

If $r > p$, we use the equality (2.13) and we get

$$(2.18) \qquad \frac{p}{p+1} \frac{d}{dt} \|v(t)\|_{p+1}^{p+1} + 2J^*(v_0) \geq \alpha \frac{r-1}{r+1} \|v(t)\|_{r+1}^{r+1}.$$

By using the Hölder inequality

$$(2.19)$$

$$\|v\|_{r+1}^{r+1} \geq C_2(\Omega) \|v\|_{p+1}^{r+1} \qquad \forall v \in L^{r+1}(\Omega) \text{ with } C_2(\Omega) = (\text{mes } \Omega)^{-\frac{r-p}{p+1}},$$

we deduce if $J^*(v_0) \leq 0$, then

$$\frac{d}{dt} \|v(t)\|_{p+1}^{-(r-p)} \leq -\alpha \frac{r-1}{r+1} \frac{r-p}{p} C_2(\Omega)$$

and by integrating in time we get

$$\|v_0\|_{p+1}^{r-p} \left[1 - \alpha \frac{r-1}{r+1} \frac{r-p}{p} C_2(\Omega) \|v_0\|_{p+1}^{r-p} t \right]^{-1} \leq \|v(t)\|_{p+1}^{r-p}.$$

The first member of this inequality becomes infinite at the time T_2. We deduce that T_2 is a bound on the maximal time of existence, and by using the same argument as Levine-Sacks ([13]), we obtain that the solution blows up at a time $T_b \leq T_2$ and from Lemma 2.1 such that $T_b \geq T_1$.

Theorem 2.5. *If $r \leq p$ and v_0 satisfies*

(2.20)
$$\alpha\|v_0\|_{p+1}^{r-1} < C(\Omega)/C_2(\Omega),$$

then the solution of (2.1) *vanishes in a finite time T_e such that*

(2.21)
$$T_{e_1} \leq T_e \leq T_{e_2}$$

with

(2.22)
$$T_{e_1} = \frac{1}{2} \frac{p}{p+1} \frac{\|v_0\|_{p+1}^{p+1}}{J^*(v_0)} \quad and \quad T_{e_2} = \frac{p}{p-1} \frac{\|v_0\|_{p+1}^{p-1}}{C(\Omega) - C_2(\Omega)\alpha\|v_0\|_{p+1}^{r-1}}.$$

Proof. From the relation (2.12) we get

$$\frac{p}{p+1} \frac{d}{dt} \|v(t)\|_{p+1}^{p+1} + C(\Omega)\|v(t)\|_{p+1}^2 \leq \alpha\|v(t)\|_{r+1}^{r+1}.$$

The right side can be bounded by using the Hölder inequality

(2.23)
$$\|v\|_{r+1}^{r+1} \leq C_2(\Omega) \|v\|_{p+1}^{r+1} \quad \forall v \in L_{p+1}(\Omega).$$

Thus by denoting $\varphi(t) = C(\Omega) - \alpha C_2(\Omega)\|v(t)\|_{p+1}^{r-1}$, we get

(2.24)
$$\frac{d}{dt} \|v(t)\|_{p+1}^{p-1} \leq -\frac{p-1}{p} \varphi(t).$$

We can check that if $\varphi(0) > 0$, namely $\alpha C_2(\Omega)\|v_0\|_{p+1}^{r-1} < C(\Omega)$, the mapping $t \longmapsto \varphi(t)$ is increasing, which in addition to (2.24) implies $\frac{d}{dt}\|v(t)\|_{p+1}^{p-1} \leq -\frac{p-1}{p}\varphi(0)$. By integrating in time we get

$$\|v(t)\|_{p+1}^{p-1} \leq \|v_0\|_{p+1}^{p-1} - \frac{p-1}{p}\varphi(0)t.$$

If the right side of this inequality becomes null in a finite time, then the solution v vanishes in a finite time T_e such that $T_e \leq T_{e_2}$.

From Theorem 2.4, the solution cannot vanish if $J^*(v_0) \leq 0$; hence the left inequality in (2.21) proceeds easily from (2.11).

Theorem 2.6. *If $r > p$ and v_0 satisfies*

(2.25)
$$\alpha\frac{r-1}{p-1}\|v_0\|_{\infty}^{r-p}\|v_0\|_{p+1}^{p-1} \leq C(\Omega),$$

the solution of (2.1) *vanishes in a finite time T_e such that*

(2.26)
$$T_{e_1} \leq T_e \leq T_{e_3}$$

with

(2.27)
$$T_{e_3} = T_1 - T_1\left[1 - \frac{\alpha(r-1)}{C(\Omega)(p-1)} \|v_0\|_{\infty}^{r-p}\|v_o\|_{p+1}^{p-1}\right]^{\frac{r-p}{r-1}}.$$

Proof. From the relation (2.12) we get

$$\frac{p}{p+1} \frac{d}{dt}\|v(t)\|_{p+1}^{p+1} + C(\Omega)\|v(t)\|_{p+1}^2 \leq \alpha\|v(t)\|_{\infty}^{r-p}\|v(t)\|_{p+1}^{p+1}.$$

Hence we deduce

$$\frac{p}{p-1} \frac{d}{dt}\|v(t)\|_{p+1}^{p-1} + C(\Omega) \leq \alpha\|v(t)\|_{\infty}^{r-p}\|v(t)\|_{p+1}^{p-1}.$$

We use the estimate (2.7) to obtain

$$\frac{d}{dt}\|v(t)\|_{p+1}^{p-1} + \frac{p-1}{p}\,C(\Omega) \le \alpha\frac{p-1}{p}\,\frac{T_1}{T_1-t}\,\|v_0\|_\infty^{r-p}\|v(t)\|_{p+1}^{p-1}.$$

Let us denote $C_1 = \frac{p-1}{p}C(\Omega)$, $\mu = \frac{p-1}{r-p}$; this inequality may be written as

$$\frac{d}{dt}\left((T_1-t)^\mu\|v(t)\|_{p+1}^{p-1}\right) \le -C_1(T_1-t)^\mu,$$

and by integrating in time between 0 and t, it follows that

(2.28)
$$\|v(t)\|_{p+1}^{p-1} \le \left[\|v_0\|_{p+1}^{p-1} - \frac{C_1T_1}{\mu+1}\right]\frac{T_1^\mu}{(T_1-t)^\mu} + \frac{C_1}{\mu+1}\,(T_1-t).$$

If v_0 satisfies $\|v_0\|_{p+1}^{p-1} \le \frac{C_1T_1}{\mu+1}$, namely if $\alpha\frac{r-1}{p-1}\|v_0\|_\infty^{r-p}\|v_0\|_{p+1}^{p-1} \le C(\Omega)$, the right side of the inequality (2.28) becomes null at the time T_{e_2} such that

$$(T_1 - T_{e_2})^{\mu+1} = T_1^{\mu+1}\left[1 - (\mu+1)\frac{\|v_0\|_{p+1}^{p-1}}{C_1T_1}\right].$$

We conclude that the function v becomes null in a finite time $T_e \le T_{e_2}$.

The left inequality of (2.26) again proceeds from (2.11).
In the particular case $r = 1$ (i.e., $p_1 = m$) we have more accurate results.
We introduce the functional F defined by

(2.29)
$$F(\varphi) = \frac{1}{\|\varphi\|_{p+1}^2}\int_\Omega\left(|\nabla\varphi|^2 - \alpha\varphi^2\right)dx = \frac{2J^*(\varphi)}{\|\varphi\|_{p+1}^2}\quad \forall\varphi \in H_0^1(\Omega).$$

Let us denote λ_1 the first eigenvalue of the Dirichlet problem $-\Delta\rho = \lambda\rho, x \in \Omega$, $\rho = 0, x \in \partial\Omega$.
We have the inequality $F(\varphi) \ge C(\Omega)\frac{\lambda_1-\alpha}{\lambda_1}$, and since

(2.30)
$$\frac{d}{dt}\|v(t)\|_{p+1}^{p-1} = -\frac{p-1}{p}F(v(t))$$

we get

(2.31)
$$\|v(t)\|_{p+1}^{p-1} \le \|v_0\|_{p+1}^{p-1} - \frac{p-1}{p}\frac{\lambda_1-\alpha}{\lambda_1}\,C(\Omega)t.$$

Then, if $\alpha < \lambda_1$, the second member becomes null in a finite time, so the solution vanishes in a finite time T_e such that

$$T_e \le \frac{p}{p-1}\frac{\lambda_1}{\lambda_1-\alpha}\frac{\|v_0\|_{p+1}^{p-1}}{C(\Omega)}.$$

We easily prove that the mapping $t \longmapsto F(v(t))$ is decreasing, then from (2.30), we get

(2.32)
$$\|v(t)\|_{p+1}^{p-1} \ge \|v_0\|_{p+1}^{p-1} - \frac{p-1}{p}F(v_0)t.$$

In the case $\lambda_1 > 0$, if $F(v_0) < 0$, then we obtain immediately that $\lim_{t\longrightarrow+\infty}\|v(t)\|_{p+1} = +\infty$; if it is not the case, by using the same argument as Friedman and McLeod in [6], we prove again that $\lim_{t\longrightarrow+\infty}\|v(t)\|_{p+1} = +\infty$ ([14]).

If $\alpha = \lambda_1$, then the problem has a global solution which tends to $\theta\rho_1$ as $t \longrightarrow +\infty$ (ρ_1 is the eigenfunction corresponding to λ_1 satisfying $\rho_1 > 0$, $\|\rho_1\|_{L^1(\Omega)} = 1$ and $\theta = (\int_\Omega v_0^p \rho_1 dx)^{\frac{1}{p}}$). The proof uses the same argument as Sacks in [18].

In the particular case $r = p$ (i.e., $p_1 = 1$) we also have more precise results (see [10]).

We define the functional G as

$$G(\varphi) = \frac{1}{\|\varphi\|_{p+1}^2} \int_\Omega |\nabla\varphi|^2 dx, \ \forall\varphi \in H_0^1(\Omega),$$

and we prove that the mapping $t \longmapsto G(v(t))$ is decreasing. From the equality

$$\frac{p}{p-1} \frac{d}{dt}\|v(t)\|_{p+1}^{p-1} + G(v(t)) = \alpha\|v(t)\|_{p+1}^{p-1}$$

we get

$$\exp\Big(-\alpha\frac{p-1}{p}t\Big)\|v(t)\|_{p+1}^{p-1} + \frac{p-1}{p}\int_0^t G(v(s))\exp\Big(-\alpha\frac{p-1}{p}s\Big)ds = \|v_0\|_{p+1}^{p-1},$$

and since the function $t \longmapsto G(v(t))$ is decreasing, we obtain

(2.33) $$\|v(t)\|_{p+1}^{p-1} \geq \exp(\alpha\frac{p-1}{p}t)\Big(\|v_0\|_{p+1}^{p-1} - \frac{G(v_0)}{\alpha}\Big) + \frac{G(v_0)}{\alpha}$$

and
(2.34)

$$\|v(t)\|_{p+1}^{p-1} \leq \exp\Big(\alpha\frac{p-1}{p}t\Big)\Big(\|v_0\|_{p+1}^{p-1} - \frac{G(v(t))}{\alpha}\Big) + \frac{G(v(t))}{\alpha}.$$

Since $G(v(t)) \geq C(\Omega)$, if $\alpha\|v_0\|_{p+1}^{p-1} < C(\Omega)$, the solution vanishes in a finite time T_e such that

$$T_e \leq \frac{p}{\alpha(p-1)} \ln\Big(\frac{C(\Omega)}{C(\Omega) - \alpha\|v_0\|_{p+1}^{p-1}}\Big).$$

If $\alpha\|v_0\|_{p+1}^{p-1} > G(v_0)$, then the solution tends to infinity as $t \longrightarrow +\infty$.

3. DEFINITION OF THE NUMERICAL SCHEME

If we use a classical Euler scheme for the semidiscretization in time of the problem, the corresponding numerical solution cannot vanish or blow up after a finite number of time steps.

So, we generalize here the numerical scheme used in [8]: if v_n is the approximate value of the solution at the time level $t_n = n\Delta t$ (Δt is the time step), then v_{n+1} is the solution of the equation

(3.1)

$$\frac{p}{p-1} v_{n+1}(v_{n+1}^{p-1} - v_n^{p-1}) + \Delta t A v_{n+1} = \Delta t(\alpha_1 v_{n+1}v_n^{r-1} + \alpha_2 v_{n+1}^p v_n^{r-p})$$

with $\alpha_1 + \alpha_2 = \alpha, \alpha_1, \alpha_2 \geq 0$.

If $r \leq p$, we choose $\alpha_2 = 0, \alpha_1 = \alpha$ in order to avoid negative powers in the second member.

If $r > p$, if $\alpha_2 = 0$, the solution of (3.1) may become null in a finite time, but it cannot blow up in a finite time when this physical peculiarity appears for the continuous problem. If $\alpha_1 = 0$, the solution may become null or infinite in a finite time, but this solution is not always bounded by the solution of the differential

equation (2.8) as it is the case for the exact solution; it depends on the values of r and p. So we choose α_1 and α_2 such that the numerical solution is bounded by the solution of the differential equation and is as close as possible to this solution.

This leads us to consider the following values of α_1 and α_2:

$$(3.2) \qquad \begin{array}{l} \alpha_1 = \alpha, \ \alpha_2 = 0 \qquad \text{if } r \leq p, \\[2mm] \alpha_1 = \dfrac{2p-1-r}{p-1}\alpha, \ \ \alpha_2 = \dfrac{r-p}{p-1}\alpha \qquad \text{if } p \leq r \leq 2p-1, \\[2mm] \alpha_1 = 0, \ \alpha_2 = \alpha \qquad \text{if } r \geq 2p-1. \end{array}$$

Lemma 3.1. *If the solution v_n is positive on Ω, $v_n \in H_0^1(\Omega) \cap C^\epsilon(\overline{\Omega}), (\epsilon \in]0,1[)$ and satisfies*

$$(3.3) \qquad \alpha_2 \frac{p-1}{p} \Delta t \|v_n\|_\infty^{r-p} < 1 \ \text{ in the case } r > p,$$

then the problem (3.1) has a maximal nonnegative solution $\overline{v} \in H_0^1(\Omega) \cap C^2(\overline{\Omega})$. Besides every nonnegative solution v satisfies $0 \leq v \leq \overline{v}$.

Proof. The problem (3.1) may be written as $Av_{n+1} = f(v_{n+1})$, where f is the function defined by

$$(3.4)$$
$$f(u) = \frac{p}{(p-1)\Delta t} \left[\left(1 + \alpha_1 \frac{p-1}{p} \Delta t v_n^{r-p} \right) v_n^{p-1} u - \left(1 - \alpha_2 \frac{p-1}{p} \Delta t v_n^{r-p} \right) u^p \right],$$

where f satisfies $f(0) = 0$.

In addition, if (3.3) is satisfied, (3.1) has a constant supersolution C_n such that

$$(3.5) \qquad C_n^{p-1} = \frac{\|v_n\|_\infty^{p-1} + \alpha_1 \frac{p-1}{p} \Delta t \|v_n\|_\infty^{r-1}}{1 - \alpha_2 \frac{p-1}{p} \Delta t \|v_n\|_\infty^{r-p}}.$$

Hence we deduce from a result from Amann [1] that (3.1) has a minimal solution (the null solution) and a maximal solution $\overline{v} \in C_2(\overline{\Omega})$. Besides any solution satisfies $v \in C^2(\overline{\Omega})$ and $0 \leq v \leq \overline{v}$.

Lemma 3.2. *Under the hypothesis of the previous lemma, (3.1) has at most one positive solution.*

The proof is the same as in [8].

In order to set up a sufficient condition for the solution of the numerical sheme to be positive, we introduce the functional F, which is defined by

$$(3.6)$$
$$F(\varphi) = \frac{1}{\|\varphi\|_{p+1}^2} \int_\Omega \left(|\nabla \varphi|^2 - \alpha_1 \varphi^{r+1} \right) dx, \ \ \forall \varphi \in H_0^1(\Omega) \cap L^{r+1}(\Omega).$$

Lemma 3.3. *Under the hypothesis of the Lemma 3.1 and if v_n satisfies*

$$(3.7) \qquad \frac{p-1}{p} \Delta t F(v_n) < \|v_n\|_{p+1}^{p-1},$$

then (3.1) has a positive solution in $H_0^1(\Omega) \cap C^2(\overline{\Omega})$.

Proof. Let us introduce the set K and the functional J_n on $H_0^1(\Omega)$

$$(3.8) \qquad K = \left\{ v \in H_0^1(\Omega) / \int_\Omega \left(1 - \alpha_2 \frac{p-1}{p} \Delta t v_n^{r-p} \right) |v|^{p+1} dx = 1 \right\},$$

(3.9)

$$J_n(v) = \int_\Omega |\nabla v|^2 dx - \frac{p}{(p-1)\Delta t} \int_\Omega v_n^{p-1} \left(1 + \alpha_1 \frac{p-1}{p} \Delta t v_n^{r-p} \right) v^2 dx,$$

and let us consider the minimization problem $\varphi_n \in K, J_n(\varphi_n) = \min_{v \in K} J_n(v)$. By using a Hölder inequality, we easily set

$$J_n(v) \geq -\frac{p}{(p-1)\Delta t} \int_\Omega v_n^{p+1} \frac{\left(1 + \alpha_1 \frac{p-1}{p} \Delta t v_n^{r-p} \right)^{\frac{p+1}{p-1}}}{\left(1 - \alpha_2 \frac{p-1}{p} \Delta t v_n^{r-p} \right)^{\frac{2}{p-1}}} dx, \ \forall v \in K.$$

Hence, J_n has a lower bound on K.

Let $\varphi_{n,k}$ be a minimizing sequence; since $J_n(|\varphi_{n,k}|) = J(\varphi_{n,k})$, we may suppose $\varphi_{n,k \geq 0}$. This sequence is bounded in $H_0^1(\Omega)$, so we can extract a subsequence, again labeled $\varphi_{n,k}$, which converges to φ_n weakly in $H_0^1(\Omega)$ and strongly in $L^{p+1}(\Omega)$. Hence we have $\varphi_n \geq 0, \varphi_n \in K$ and for all $v \in H_0^1(\Omega)$, we obtain

(3.10)

$$A\varphi_n - \left(\frac{p}{(p-1)\Delta t} v_n^{p-1} + \alpha_1 v_n^{r-1} \right) \varphi_n = J_n(\varphi_n) \left(1 - \alpha_2 \frac{p-1}{p} \Delta t v_n^{r-p} \right) \varphi_n^p.$$

Then, if $J_n(\varphi_n) < 0$, the solution v_{n+1} of the problem (3.1) is defined by

$$(3.11) \qquad v_{n+1} = \left(-\frac{p-1}{p} \Delta t J_n(\varphi_n) \right)^{\frac{1}{p-1}} \varphi_n.$$

It remains to determine a sufficient condition for $J_n(\varphi_n) < 0$.

Since the function $\psi_n = (\int_\Omega (v_n^{p+1} - \alpha_2 \frac{p-1}{p} \Delta t v_n^{r+1}) dx)^{-\frac{1}{p+1}} v_n$ belongs to K, we necessarily have $J_n(\varphi_n) \leq J_n(\psi_n)$, where

(3.12)

$$J_n(\psi_n) = \|v_n\|_{p+1}^2 \left(F(v_n) - \frac{p}{(p-1)\Delta t} \|v_n\|_{p+1}^{p-1} \right)$$

$$\times \left(\int_\Omega \left(v_n^{p+1} - \alpha_2 \frac{p-1}{p} \Delta t v_n^{r+1} \right) dx \right)^{-\frac{2}{p+1}}.$$

Hence if $J_n(\psi_n) < 0$, namely if $\frac{p-1}{p} \Delta t F(v_n) < \|v_n\|_{p+1}^{p-1}$, then we obtain $J_n(\varphi_n) < 0$ and (3.1) admits one positive solution.

Now we prove that the numerical solution is bounded by one of the differential equations (2.8), as it is true for the exact solution.

Lemma 3.4. *If the parameters α_1 and α_2 are chosen as in (3.2) and if the hypotheses of Lemmas 3.1 and 3.3 are satisfied, we have the inequality*

$$(3.13) \qquad \|v_n\|_\infty \leq \|v_0\|_\infty \left(1 + \alpha \frac{p-r}{p} \|v_0\|_\infty^{r-p} t_n \right)^{\frac{1}{p-r}} \quad \text{if } r \neq p,$$

(3.14) $$\|v_n\|_\infty \leq \|v_0\|_\infty \exp\left(\frac{\alpha}{p} t_n\right).$$

Proof. We denote by w_n the solution of the differential equation (2.8) at the time level $t_n = n\Delta t$,

$$w_n = w_0\left(1 + \alpha\frac{p-r}{p}\Delta t w_0^{r-p}\right)^{\frac{1}{p-r}} \quad \text{if } r \neq p,$$

$$w_n = w_0 \exp\left(\frac{\alpha}{p} t_n\right),$$

and we prove the lemma recurrently: the result is true for $n = 0$ since $w(0) = \|v_0\|_\infty$. Then w_{n+1} is a supersolution of (3.1) if $f(w_{n+1}) \leq 0$; namely

$$v_n^{p-1}\left(1 + \alpha_1\frac{p-1}{p}\Delta t v_n^{r-p}\right) \leq w_{n+1}^{p-1}\left(1 - \alpha_2\frac{p-1}{p}\Delta t v_n^{r-p}\right),$$

that is

$$v_n^{p-1}\left(1 + \alpha_1\frac{p-1}{p}\Delta t v_n^{r-p}\right)$$

$$\leq w_n^{p-1}\left(1 + \alpha\frac{p-r}{p}\Delta t w_n^{r-p}\right)^{\frac{p-1}{p-r}}\left(1 - \alpha_2\frac{p-1}{p}\Delta t v_n^{r-p}\right) \quad \text{if } r \neq p$$

and

$$v_n^{p-1}\left(1 + \alpha\frac{p-1}{p}\Delta t\right) \leq w_n^{p-1} \exp\left(\frac{\alpha(p-1)}{p}\Delta t\right) \quad \text{if } r = p.$$

As $\|v_n\|_\infty \leq w_n$, in the case $r = p$, the above inequality is immediately verified and in the case $r \neq p$ it will be true if

(3.15)

$$1 + \alpha_1\frac{p-1}{p}\Delta t w_n^{r-p} \leq \left(1 - \alpha_2\frac{p-1}{p}\Delta t w_n^{r-p}\right)\left(1 + \alpha\frac{p-r}{p}\Delta t w_n^{r-p}\right)^{\frac{p-1}{p-r}}$$

Let us prove that this last inequality holds if α_1 and α_2 are chosen as in (3.2). For this, we consider the mapping $h : x \longmapsto (1 - \alpha_2 x)(1 + \alpha\frac{x}{\mu})$, where $\mu = \frac{p-1}{p-r}, r \neq p$. This function is defined for $x \geq 0$ if $r < p$ and for $-\alpha\frac{x}{\mu} \leq 1$ if $r > p$. Moreover, it is easy to check that $h''(x) \geq 0 \; \forall x \in [0, \frac{-\mu}{\alpha}[$. So we have the inequality $h(0) + xh'(0) \leq h(x)$. In particular, for $x = \frac{p-1}{p}\Delta t w_n^{r-p}$, we obtain the inequality (3.15). We deduce $\|v_{n+1}\|_\infty \leq w_{n+1}$, which achieves the proof.

Computation of the positive solution of the numerical scheme. Equation (3.1) may be written as

$$Av_{n+1} = f(v_{n+1}),$$

where f is the function defined in (3.4).

In order to compute a numerical solution of (3.1), we use a result of Keller [7]. The function f satisfies the inequality $f(v) - f(u) \geq -m(v - u)$, where m is the function defined on Ω by

(3.16)

$$m = \frac{p}{(p-1)\Delta t}\left(pC_n^{p-1}\left(1 - \alpha_2\frac{p-1}{p}\Delta t v_n^{r-p}\right) - v_n^{p-1}\left(1 + \alpha_1\frac{p-1}{p}\Delta t v_n^{r-p}\right)\right)$$

with C_n defined in (3.5).

So if we define the sequence $(v_{n+1,j})_{j\geq 0}$ by

(3.17) $$(A + mI)v_{n+1,j+1} = f(v_{n+1,j}) + mv_{n+1,j}$$

or

(3.18)

$$(A + mI)v_{n+1,j+1} = \frac{p}{(p-1)\Delta t}\left(1 - \alpha_2\frac{p-1}{p}\Delta t v_n^{r-p}\right)\left(pC_n^{p-1} - v_{n+1,j}^{p-1}\right)v_{n+1,j},$$

we obtain a monotone sequence if the first iterate is a subsolution or a supersolution; it is the choice of $v_{n+1,0}$ that determines whether the sequence is decreasing or increasing. This sequence converges to the solution. For example, we can choose $v_{n+1,0} = C_n$ since C_n is a supersolution and the sequence $(v_{n+1,j})_{j\geq 0}$ is a decreasing sequence converging to v_{n+1}.

If $r > p$, the constant C_n may be very large when t_n is close to T_1. So we shall prove that in this case we again obtain a convergent sequence $(v_{n+1,j})_{j\geq 0}$ by choosing $v_{n+1,0} = v_n$.

In order to prove this result, we need the following lemma.

Lemma 3.5. *For $n \geq 1$, we have the inequality*

(3.19) $$v_{n+1} \leq \left(\frac{2 + \alpha_1\frac{p-1}{p}\Delta t\|v_n\|_\infty^{r-p}}{1 - \alpha_2\frac{p-1}{p}\Delta t\|v_n\|_\infty^{r-p}}\right)^{\frac{1}{p-1}} v_n \quad if\ r \geq\ p.$$

Proof. Let $\tau \geq 1$; the function $\tau^{\frac{1}{p-1}}v_n$ will be a supersolution of (3.1) if

$$Av_n \geq \frac{p}{(p-1)\Delta t}\left(\left(v_n^p + \alpha_1\frac{p-1}{p}\Delta t v_n^r\right) - \tau\left(v_n^p - \alpha_2\frac{p-1}{p}\Delta t v_n^r\right)\right),$$

but

$$Av_n = \frac{p}{(p-1)\Delta t}\left(v_{n-1}^{p-1}\left(1 + \alpha_1\frac{p-1}{p}\Delta t v_{n-1}^{r-p}\right)v_n - \left(1 - \alpha_2\frac{p-1}{p}\Delta t v_{n-1}^{r-p}\right)v_n^p\right),$$

which implies $Av_n \geq -\frac{p}{(p-1)\Delta t}v_n^p$. Then for $n \geq 1$, the inequality will hold if

$$\frac{p}{(p-1)\Delta t}\ (\tau - 2)v_n^p \geq \alpha_1 v_n^r + \tau\alpha_2 v_n^r,$$

which will be true if

$$\tau \geq \frac{2 + \alpha_1\frac{p-1}{p}\Delta t\|v_n\|_\infty^{r-p}}{1 - \alpha_2\frac{p-1}{p}\Delta t\|v_n\|_\infty^{r-p}}.$$

Theorem 3.6. *If $r \geq p$ and $v_{n+1,0} = v_n(n \geq 1)$, the sequence $(v_{n+1,j})_{j\geq 0}$ converges uniformly to v_{n+1}.*

Proof. Let us consider the sequences $(w_j)_{j\geq 0}, (z_j)_{j\geq 0}$ obtained from (3.18) with the following respective initial values:

$$w_0 = \left(\frac{1 - \alpha_2\frac{p-1}{p}\Delta t\|v_n\|_\infty^{r-p}}{2 + \alpha_1\frac{p-1}{p}\Delta t\|v_n\|_\infty^{r-p}}\right)^{\frac{1}{p-1}} v_{n+1}, \quad z_0 = C_n.$$

The following inequalities hold: $w_0 \leq v_n \leq z_0$ (see (3.19)). Since the operator $(A + mI)$ is monotone and the mapping $u \longmapsto f(u) + mu$ is increasing, we deduce that $w_j \leq v_{n+1,j} \leq z_j$.

Besides, since w_0 is a subsolution of (3.1), the sequence $(w_j)_{j \geq 0}$ is a monotone increasing sequence converging uniformly to v_{n+1}. In the same way, since z_0 is a supersolution of (3.1), the sequence $(z_j)_{j \geq 0}$ is a monotone decreasing sequence converging uniformly to v_{n+1}. Therefore, the sequence $(w_j)_{j \geq 0}$ converges uniformly to v_{n+1}.

Remark 3.7. We also observe that the sequence $(v_{n+1,j})_{j \geq 0}$ converges more rapidly by taking $v_{n+1,0} = v_n$ in the case $r < p$.

4. PROPERTIES OF THE NUMERICAL SOLUTION

In this part, we check that the numerical solution has the same properties as the exact solution. We prove that the solution of the numerical scheme exists and remains positive during at least a finite lapse of time and we obtain sufficient conditions for this solution to become infinite or null in a finite time.

Lemma 4.1. *For $n \geq 0$, we have the inequality*

(4.1)
$$\frac{p-1}{p}\Delta t \frac{1}{\|v_{n+1}\|_{p+1}^2} \int_\Omega \left(|\nabla v_{n+1}|^2 - \alpha_2 v_n^{r-p} v_{n+1}^{p+1} - \alpha_1 v_n^{r-1} v_{n+1}^2 \right) dx$$
$$\leq \|v_n\|_{p+1}^{p-1} - \|v_{n+1}\|_{p+1}^{p-1}.$$

Proof. By multiplying equation (3.1) by v_{n+1} and integrating on Ω, we get

$$\frac{p}{p-1} \int_\Omega v_{n+1}^2 \left(v_{n+1}^{p-1} - v_n^{p-1} \right) dx + \Delta t \int_\Omega |\nabla v_{n+1}|^2 dx$$
$$= \Delta t \left(\alpha_1 \int_\Omega v_n^{r-1} v_{n+1}^2 dx + \alpha_2 \int_\Omega v_n^{r-p} v_{n+1}^{p+1} dx \right).$$

Besides, we have the inequality

$$\int_\Omega v_{n+1}^2 \left(v_{n+1}^{p-1} - v_n^{p-1} \right) dx \geq \|v_{n+1}\|_{p+1}^2 \left(\|v_{n+1}\|_{p+1}^{p-1} - \|v_n\|_{p+1}^{p-1} \right).$$

We deduce the result.

Lemma 4.2. *For $n \geq 0$, we have the inequalities.*
   If $r \leq p$,

(4.2)
$$\|v_n\|_{p+1}^{p-1} - \|v_{n+1}\|_{p+1}^{p-1} \leq \frac{p-1}{p}\Delta t \frac{1}{\|v_n\|_{p+1}^2} \int_\Omega \left(|\nabla v_n|^2 - \alpha v_n^{r+1} \right) dx.$$

   If $r > p$,

(4.3)
$$\|v_n\|_{p+1}^{p+1} - \|v_{n+1}\|_{p+1}^{p+1}$$
$$\leq \frac{p+1}{p}\Delta t \left(\int_\Omega \left(|\nabla v_n|^2 - \alpha v_n^{r+1} \right) dx + \alpha_2 \frac{p-1}{p+1} \int_\Omega v_n^{r-p} \left(v_n^{p+1} - v_{n+1}^{p+1} \right) dx \right).$$

Proof. From Lemma 3.3, the solution of (3.1) is written as

$$v_{n+1} = \left(-\frac{p-1}{p}\Delta t J_n(\varphi_n)\right)^{\frac{1}{p-1}}\varphi_n.$$

Moreover, the function φ_n as an element of K satisfies

$$\int_\Omega \left(1 - \alpha_2 \frac{p-1}{p}\Delta t v_n^{r-p}\right)\varphi_n^{p+1}dx = 1.$$

With both the previous arguments, we obtain

$$\left(\int_\Omega \left(1 - \alpha_2 \frac{p-1}{p}\Delta t v_n^{r-p}\right)v_{n+1}^{p+1}dx\right)^{\frac{p-1}{p+1}} = -\frac{p-1}{p}\Delta t J_n(\varphi_n).$$

Besides, from (3.12) we get

$$J_n(\varphi_n) \leq J_n(v_n)\left(\int_\Omega \left(1 - \alpha_2\frac{p-1}{p}\Delta t v_n^{r-p}\right)v_n^{p+1}dx\right)^{-\frac{2}{p+1}},$$

and we obtain

(4.4)
$$-\frac{p-1}{p}\Delta t J_n(v_n) \leq \left(\int_\Omega \left(1 - \alpha_2\frac{p-1}{p}\Delta t v_n^{r-p}\right)v_n^{p+1}dx\right)^{\frac{2}{p+1}}$$
$$\times \left(\int_\Omega \left(1 - \alpha_2\frac{p-1}{p}\Delta t v_n^{r-p}\right)v_{n+1}^{p+1}dx\right)^{\frac{p-1}{p+1}}.$$

If $r \leq p$, then $\alpha_2 = 0$ and we get

$$-\frac{p-1}{p}\Delta t J_n(v_n) \leq \|v_n\|_{p+1}^2\|v_{n+1}\|_{p+1}^{p-1},$$

since

$$J_n(v_n) = \int_\Omega (|\nabla v_n|^2 - \alpha v_n^{r+1})dx - \frac{p}{(p-1)\Delta t}\|v_n\|_{p+1}^{p+1},$$

we deduce (4.2).

If $r > p$, then $\alpha_2 > 0$ and by using the Young inequality, we get

$$-\frac{p-1}{p}\Delta t J_n(v_n) \leq \frac{2}{p+1}\int_\Omega \left(1 - \alpha_2\frac{p-1}{p}\Delta t v_n^{r-p}\right)v_n^{p+1}dx$$
$$+ \frac{p-1}{p+1}\int_\Omega \left(1 - \alpha_2\frac{p-1}{p}\Delta t v_n^{r-p}\right)v_{n+1}^{p+1}dx,$$

and by using the definition of J_n, we obtain

$$-\frac{p-1}{p}\Delta t\int_\Omega \left(|\nabla v_n|^2 - \alpha_1 v_n^{r+1}\right)dx \leq \frac{p-1}{p+1}\left(-\|v_n\|_{p+1}^{p+1} + \|v_{n+1}\|_{p+1}^{p+1}\right)$$
$$-\frac{p-1}{p}\Delta t\alpha_2\int_\Omega \left(\frac{2}{p+1}v_n^{p+1} + \frac{p-1}{p+1}v_{n+1}^{p+1}\right)v_n^{r-p}dx,$$

which gives the result. $\qquad\square$

Lemma 4.3. *For $n \geq 0$, we have*

(4.5) $$\int_\Omega v_n^{p-1}(v_{n+1} - v_n)^2 dx \leq 2\frac{p+1}{p}\,\Delta t(J^*(v_n) - J^*(v_{n+1})).$$

Proof. From the following inequality shown in [8],

$$a^{p-1}(b-a)^2 \le a^{p+1} - b^{p+1} + \frac{p+1}{p-1}b^2(b^{p-1} - a^{p-1}),$$

we deduce that

$$\int_\Omega v_n^{p-1}(v_{n+1} - v_n)^2 dx \le \|v_n\|_{p+1}^{p+1} - \|v_{n+1}\|_{p+1}^{p+1} + \frac{p+1}{p-1}\int_\Omega v_{n+1}^2(v_{n+1}^{p-1} - v_n^{p-1})dx.$$

Besides, we have

$$\int_\Omega v_{n+1}^2(v_{n+1}^{p-1} - v_n^{p-1})dx$$

$$= \frac{p-1}{p}\Delta t\left(-\int_\Omega \left(|\nabla v_{n+1}|^2 - \alpha_1 v_{n+1}^2 v_n^{r-1} - \alpha_2 v_{n+1}^{p+1} v_n^{r-p}\right)dx\right).$$

Therefore, the previous inequality becomes

$$\int_\Omega v_n^{p-1}(v_{n+1} - v_n)^2 dx \le \|v_n\|_{p+1}^{p+1} - \|v_{n+1}\|_{p+1}^{p+1}$$

$$+ \frac{p+1}{p}\Delta t\left(-\int_\Omega (|\nabla v_{n+1}|^2 - \alpha_1 v_{n+1}^2 v_n^{r-1} - \alpha_2 v_{n+1}^{p+1} v_n^{r-p})dx\right).$$

If $r \le p$, we have the inequality

$$\|v_n\|_{p+1}^{p+1} - \|v_{n+1}\|_{p+1}^{p+1} \le \frac{p+1}{p-1}\|v_n\|_{p+1}^2(\|v_n\|_{p+1}^{p-1} - \|v_{n+1}\|_{p+1}^{p-1}),$$

and from (4.2), we get

$$\|v_n\|_{p+1}^{p+1} - \|v_{n+1}\|_{p+1}^{p+1} \le \frac{p+1}{p}\Delta t\int_\Omega(|\nabla v_n|^2 - \alpha v_n^{r+1})dx.$$

If $r > p$, we use the inequality (4.3) directly and we obtain $r > p$ and $r \le p$

$$\int_\Omega v_n^{p-1}(v_{n+1} - v_n)^2 dx \le \frac{p+1}{p}\Delta t\int_\Omega \left(|\nabla v_n|^2 - \alpha v_n^{r+1}\right)dx$$

$$+ \alpha_2\frac{p-1}{p}\Delta t\int_\Omega v_n^{r-p}\left(v_n^{p+1} - v_{n+1}^{p+1}\right)dx$$

$$- \frac{p+1}{p}\Delta t\int_\Omega \left(|\nabla v_{n+1}|^2 - \alpha_1 v_{n+1}^2 v_n^{r-1} - \alpha_2 v_{n+1}^{p+1} v_n^{r-p}\right)dx.$$

By using the Young inequality, we easily prove that

$$\int_\Omega v_n^{p-1}(v_{n+1} - v_n)^2 dx \le 2\frac{p+1}{p}\Delta t\left(J^*(v_n) - J^*(v_{n+1})\right),$$

and we deduce that the sequence $(J^*(v_n))_{n\ge0}$ is nonincreasing.

Now we state an existence theorem concerning the positive solution of the numerical scheme.

Theorem 4.4. *In the case $r \le p$, if $J^*(v_0) \le 0$, then (3.1) has for $n \ge 0$ a positive solution $v_{n+1} \in H_0^1(\Omega) \cap C^2(\overline{\Omega})$. If $J^*(v_0) > 0$, then this problem has a positive solution at least until the time T_{e_1} defined in (2.22); this solution can become null after this time.*

Proof. For $n \geq 0$, (3.1) has a maximal solution. We prove that this solution is positive and then unique during a nonempty lapse of time. From (4.2), we have

$$(4.6) \qquad \|v_n\|_{p+1}^{p+1} - \|v_{n+1}\|_{p+1}^{p+1} \leq \frac{p+1}{p} \Delta t \|v_n\|_{p+1}^2 F(v_n)$$

and $F(v_n)\|v_n\|_{p+1}^2 \leq 2J^*(v_n) - \alpha \frac{r-1}{r+1} \|v_n\|_{r+1}^{r+1}$.

Since the sequence $(J^*(v_n))_{n \geq 0}$ is decreasing, we get $F(v_n)\|v_n\|_{p+1}^2 \leq 2J^*(v_0)$ and we deduce

$$(4.7) \qquad \|v_0\|_{p+1}^{p+1} - 2\frac{p+1}{p} J^*(v_0) t_{n+1} \leq \|v_{n+1}\|_{p+1}^{p+1}.$$

So, if $J^*(v_0) \leq 0$, the solution $(v_n)_{n \geq 0}$ is positive for any $n \geq 0$, and if $J^*(v_0)$ is positive, it is positive at least until the time T_{e_1}.

We deduce from Lemma 3.4 that this solution is always bounded.

Theorem 4.5. *In the case $r > p$, if $J^*(v_0) \leq 0$, then (3.1) has a positive solution for $n \geq 0$ at least until the time T_1 defined in (2.6).*

If $J^*(v_0) > 0$, (3.1) has a positive solution during a nonempty lapse of time; this solution may become null only after the time T_{e_1}, or blow up only after the time T_1.

Proof. In this case, from Lemma 3.4, (3.1) has a bounded solution at least until the time T_1 and from the inequality (4.3), we get

$$\|v_n\|_{p+1}^{p+1} - \|v_{n+1}\|_{p+1}^{p+1}$$
$$\leq \frac{p+1}{p} \Delta t \left(J^*(v_n) - \left(\alpha_1 \frac{r-1}{r+1} + \alpha_2 \left(\frac{r-1}{r+1} - \frac{p-1}{p+1} \right) \right) \int_\Omega v_n^{r+1} dx \right)$$
$$- \alpha_2 \frac{p-1}{p} \Delta t \int_\Omega v_n^{r-p} v_{n+1}^{p+1} dx,$$

and again we obtain

$$\|v_n\|_{p+1}^{p+1} - \|v_{n+1}\|_{p+1}^{p+1} \leq 2\frac{p+1}{p} \Delta t J^*(v_n).$$

Then the inequality (4.7) holds and we conclude as in the previous theorem.

Theorem 4.6. *If $r \leq p$ and $J^*(v_0) \leq 0$, then $\lim_{n \to +\infty} \|v_n\|_{p+1} = +\infty$, (except in the case $r = 1$ and $J^*(v_0) = 0$).*

The proof of this theorem is analogous to the proof of Theorem 2.4.

Theorem 4.7. *If $r > p$ and $J^*(v_0) \leq 0$, then the solution of the numerical scheme blows up in a finite time T_b^* such that*

$$(4.8) \qquad T_1 \leq T_b^* \leq T_2' \quad \text{with } T_2' \geq T_2 \text{ defined in (2.15)}.$$

Proof. According to the definition of J_n in Lemma 3.3, we have

$$\frac{p-1}{p} \Delta t J_n(v_n) = -\|v_n\|_{p+1}^{p+1} + \frac{p-1}{p} \Delta t \left(2J^*(v_n) + \left(\alpha_2 - \frac{r-1}{r+1} \alpha \right) \|v_n\|_{r+1}^{r+1} \right).$$

Since the sequence $(J^*(v_n))_{n \geq 0}$ is decreasing, if $J^*(v_0) \leq 0$, then we get $J^*(v_n) \leq 0$; hence

$$(4.9) \qquad \|v_n\|_{p+1}^{p+1} + \left(\frac{r-1}{r+1} \alpha - \alpha_2 \right) \frac{p-1}{p} \Delta t \|v_n\|_{r+1}^{r+1} \leq -\frac{p-1}{p} \Delta t J_n(v_n).$$

From the inequality (4.4) we have

(4.10)
$$-\frac{p-1}{p}\Delta t J_n(v_n) \leq \|v_{n+1}\|_{p+1}^{p-1}\left(\int_\Omega \left(1 - \alpha_2\frac{p-1}{p}\Delta t v_n^{r-p}\right)v_n^{p+1}dx\right)^{\frac{2}{p+1}}.$$

Therefore, by using this estimate in (4.9), we obtain

$$\|v_n\|_{p+1}^{p+1} + \left(\frac{r-1}{r+1}\alpha - \alpha_2\right)\frac{p-1}{p}\Delta t\|v_n\|_{r+1}^{r+1}$$

$$\leq \|v_{n+1}\|_{p+1}^{p-1}\left(\|v_n\|_{p+1}^{p+1} - \alpha_2\frac{p-1}{p}\Delta t\|v_n\|_{r+1}^{r+1}\right)^{\frac{2}{p+1}};$$

hence we get

$$\|v_n\|_{p+1}^{p-1}\frac{1 - \left(\frac{r-1}{r+1}\alpha - \alpha_2\right)\frac{p-1}{p}\Delta t\frac{\|v_n\|_{r+1}^{r+1}}{\|v_n\|_{p+1}^{p+1}}}{\left(1 - \alpha_2\frac{p-1}{p}\Delta t\frac{\|v_n\|_{r+1}^{r+1}}{\|v_n\|_{p+1}^{p+1}}\right)^{\frac{2}{p+1}}} \leq \|v_{n+1}\|_{p+1}^{p-1}.$$

Besides, by taking into account the inequality $(1 - qx)^{-1} \leq (1 - x)^{-q}$ $(0 \leq q \leq 1,$ $0 \leq x \leq 1)$ and using a Holder inequality, we deduce that

(4.11)
$$\|v_n\|_{p+1}^{p-1}\frac{1 + \left(\frac{r-1}{r+1}\alpha - \alpha_2\right)\frac{p-1}{p}\Delta t C_2(\Omega)\|v_n\|_{p+1}^{r-p}}{1 - \frac{2}{p+1}\alpha_2\frac{p-1}{p}\Delta t C_2(\Omega)\|v_n\|_{p+1}^{r-p}} \leq \|v_{n+1}\|_{p+1}^{p-1}.$$

In order to simplify the notations, let us denote

$$\gamma = \frac{2}{p+1}\frac{p-1}{p}\alpha_2 C_2(\Omega),$$

$$\beta = \left(\frac{r-1}{r+1}\alpha - \alpha_2\right)C_2(\Omega)\frac{p-1}{p},$$

$$\delta = C_2(\Omega)\frac{p-1}{p}, z_n = \|v_n\|_{p+1}.$$

It is easy to verify that the quantity $(\gamma + \beta)$ is positive. From (4.11), the sequence $(z_n)_{n \geq 0}$ is increasing and satisfies

$$(\gamma + \beta)\Delta t\frac{z_n^{r-1}}{1 - \gamma\Delta t z_n^{r-p}} \leq z_{n+1}^{p-1} - z_n^{p-1}.$$

Otherwise, for $a, b > 0$, we have the inequality (see [14])

(4.12)
$$b^{p-1} - a^{p-1} \leq \frac{p-1}{r-p}b^{r-1}\left(a^{p-r} - b^{p-r}\right).$$

Hence we obtain

$$\frac{r-p}{p-1}(\gamma + \beta)\Delta t\frac{z_n^{r-1}z_{n+1}^{1-r}}{1 - \gamma\Delta t z_n^{r-p}} \leq z_n^{p-r} - z_{n+1}^{p-r};$$

namely,

$$\gamma\Delta t + \left((\gamma + \beta)\frac{r-p}{p-1}\left(\frac{z_n}{z_{n+1}}\right)^{r-1} - \gamma\left(\frac{z_n}{z_{n+1}}\right)^{r-p}\right)\Delta t \leq z_n^{p-r} - z_{n+1}^{p-r}.$$

Let us consider the function $f : x \longmapsto (\gamma + \beta)\frac{r-p}{p-1}x^{r-1} - \gamma x^{r-p}$. The previous inequality is written

$$(4.13) \qquad \gamma \Delta t + f(\frac{z_n}{z_{n+1}})\Delta t \leq z_n^{p-r} - z_{n+1}^{p-r}.$$

The function f achieves its minimum at the point x_0 such that $x_0^{p-1} = \frac{p-1}{r-1}\frac{\gamma}{\gamma+\beta}$ and is decreasing on the interval $[0, x_0]$, then we shall use the estimates, $f(\frac{z_n}{z_{n+1}}) \geq f(1)$ if $x_0 \geq 1$, and $f(\frac{z_n}{z_{n+1}}) \geq f(x_0)$ if $x_0 < 1$.

If $x_0 < 1$, we obtain $\gamma\frac{r-p}{r-1}\Delta t \leq z_n^{p-r} - z_{n+1}^{p-r}$; if $x_0 \geq 1$, we get $(\gamma + \beta)\frac{r-p}{p-1}\Delta t \leq z_n^{p-r} - z_{n+1}^{p-r}$.

It remains to know the sign of $(x_0 - 1)$ according to r and p. Let $r_0 \in [p, 2p - 1]$ be the unique root of the equation $(r_0 + 1)^3 = 2(p + 1)(r_0^2 + 1)$. We easily get the following results.

- If $p \leq r \leq r_0$, then $x_0 < 1$ and we obtain

$$(4.14) \qquad \|v_n\|_{p+1}^{r-p} \geq \|v_0\|_{p+1}^{r-p}\left(\frac{T_2'}{T_2' - t_n}\right)$$

with $T_2' = \frac{(p+1)(r-1)^2}{2(r-p)(r+1)}T_2$; we conclude as in Theorem 2.4.
- If $r_0 < r \leq 2p - 1$, then $x_0 > 1$ and the inequality (4.14) holds with $T_2' = \frac{(r-1)(p+1)}{p^2-1-(r-p)^2}T_2$.
- If $2p - 1 < r \leq p + \sqrt{p^2 - 1}$, then $x_0 \geq 1$ and (4.14) holds with $T_2' = \frac{(r-1)(p+1)}{2(r-p)}T_2$.
- If $r \geq p + \sqrt{p^2 - 1}$, then $x_0 < 1$ and (4.14) holds with $T_2' = \frac{(r-1)^2(p+1)}{2(r+1)(p-1)}T_2$.

Thus we obtain an upper bound on the blow up time according the values of r and p and we can check that $T_2' \geq T_2$ in all cases.

Remark 4.8. The difference between the times T_2 and T_2' proceeds from the upper bound of the second member of the inequality (4.4) we used to obtain (4.10).

We now set up a sufficient condition on the initial data for the numerical solution to vanish in a finite time.

Theorem 4.9. *If $r \leq p$ and if v_0 satisfies (2.20), then the solution of (3.1) vanishes in a finite time T_e^* such that $T_{e_1} \leq T_e^* \leq T_{e_2}$ with T_{e_2} defined in (2.22).*

The proof of this theorem is analogous to that of Theorem 2.5.

Theorem 4.10. *If $p < r < 2p - 1$ and if v_0 satisfies*

$$(4.15) \qquad \alpha\frac{r-1}{p-1}\|v_0\|_{p+1}^{p-1}\|v_0\|_{\infty}^{r-p} \leq \left(1 - \alpha\frac{p-1}{p}\|v_0\|_{\infty}^{r-p}\Delta t\right)C(\Omega),$$

then the solution of (3.1) vanishes in a finite time T_e^* such that $T_{e_1} \leq T_e^* \leq T_{e_3}'$. And, if there exists a positive constant δ_1 such that

$$(4.16) \qquad \alpha\frac{r-1}{p-1}\|v_0\|_{\infty}^{r-p}\|v_0\|_{p+1}^{p-1} \leq C(\Omega)(1 - \delta_1),$$

then we have the estimate $T_{e_3} - T_{e_3}' = O(\Delta t)$ with T_{e_3} defined in (2.27).

Proof. The inequality (4.1) may be written as

$$\|v_{n+1}\|_{p+1}^{p-1} - \|v_n\|_{p+1}^{p-1}$$
$$\leq \frac{p-1}{p}\Delta t \frac{1}{\|v_{n+1}\|_{p+1}^2}$$
$$\times \left(-\int_\Omega |\nabla v_{n+1}|^2 dx + \alpha_2 \int_\Omega v_n^{r-p} v_{n+1}^{p+1} dx + \alpha_1 \int_\Omega v_n^{r-1} v_{n+1}^2 dx\right)$$

and by using the Sobolev constant $C(\Omega)$ we easily get

$$\|v_{n+1}\|_{p+1}^{p-1} - \|v_n\|_{p+1}^{p-1}$$
$$\leq \Delta t \frac{p-1}{p}\left(-C(\Omega) + \alpha_2 \|v_n\|_\infty^{r-p} \|v_{n+1}\|_{p+1}^{p-1} + \alpha_1 \|v_n\|_\infty^{r-p} \|v_{n+1}\|_{p+1}^{p-1}\right);$$

namely

$$\|v_{n+1}\|_{p+1}^{p-1}\left(1 - \alpha_2 \frac{p-1}{p}\Delta t \|v_n\|_\infty^{r-p}\right)$$
$$\leq \|v_n\|_{p+1}^{p-1}\left(1 + \alpha_1 \frac{p-1}{p}\Delta t \|v_n\|_\infty^{r-p}\right) - \frac{p-1}{p}\Delta t C(\Omega).$$

From Lemma 3.4, we have $\|v_n\|_\infty^{r-p} \leq \frac{T_1}{T_1-t_n}\|v_0\|_\infty^{r-p}$; that is, $\|v_n\|_\infty^{r-p} \leq \frac{p}{r-p}\frac{1}{\alpha}\frac{1}{T_1-t_n}$. Hence we obtain

$$\|v_{n+1}\|_{p+1}^{p-1}\left(1 - \frac{\alpha_2}{\alpha}\frac{p-1}{r-p}\Delta t \frac{1}{T_1-t_n}\right)$$
$$\leq \|v_n\|_{p+1}^{p-1}\left(1 + \frac{\alpha_1}{\alpha}\frac{p-1}{r-p}\Delta t \frac{1}{T_1-t_n}\right) - \frac{p-1}{p}\Delta t C(\Omega).$$

By using the notations $C_1 = \frac{p-1}{p}C(\Omega)$ and $\mu = \frac{p-1}{r-p}$, the previous inequality becomes
(4.17)

$$\|v_{n+1}\|_{p+1}^{p-1}\left(T_1 - t_n - \frac{\alpha_2}{\alpha}\mu\Delta t\right) \leq \|v_n\|_{p+1}^{p-1}\left(T_1 - t_n + \frac{\alpha_1}{\alpha}\mu\Delta t\right) - \Delta t C_1(T_1 - t_n).$$

In the case $r < 2p-1$, we have $\frac{\alpha_2}{\alpha} = \frac{1}{\mu}$ and $\frac{\alpha_1}{\alpha} = \frac{\mu-1}{\mu}$, and by multiplying (4.17) by $(T_1 - t_{n+1})^{\mu-1}$, we obtain

$$(T_1 - t_{n+1})^\mu \|v_{n+1}\|_{p+1}^{p-1}$$
(4.18)
$$\leq \|v_n\|_{p+1}^{p-1}(T_1 - t_n + (\mu-1)\Delta t)(T_1 - t_{n+1})^{\mu-1}$$
$$- \Delta t C_1(T_1 - t_n)(T_1 - t_{n+1})^{\mu-1}.$$

Since $\mu \geq 1$, we get $(T_1 - t_n + (\mu-1)\Delta t)(T_1 - t_{n+1})^{\mu-1} \leq (T_1 - t_n)^\mu$ and we obtain

$$\|v_{n+1}\|_{p+1}^{p-1}(T_1 - t_{n+1})^\mu \leq \|v_n\|_{p+1}^{p-1}(T_1 - t_n)^\mu - C_1 \Delta t (T_1 - t_n)(T_1 - t_{n+1})^{\mu-1},$$

which implies

$$\|v_{n+1}\|_{p+1}^{p-1}(T_1 - t_{n+1})^\mu \leq \|v_0\|_{p+1}^{p-1}T_1^\mu - C_1 \Delta t \sum_{j=0}^n (T_1 - t_j)(T_1 - t_{j+1})^{\mu-1}.$$

Since the mapping $t \longmapsto (T_1 - t + \Delta t)(T_1 - t)^{\mu-1}$ is decreasing, we have

$$\Delta t \sum_{j=0}^n (T_1 - t_j)(T_1 - t_{j+1})^{\mu-1} \geq \int_{\Delta t}^{t_{n+2}} (T_1 - s + \Delta t)(T_1 - s)^{\mu-1} ds,$$

that is

$$\Delta t \sum_{j=0}^{n} (T - t_j)(T_1 - t_{j+1})^{\mu-1}$$

$$\geq \frac{1}{\mu + 1}\left((T_1 - \Delta t)^{\mu}\left(T_1 + \frac{\Delta t}{\mu}\right) - (T_1 - t_{n+2})^{\mu}\left(T_1 - t_{n+1} + \frac{\Delta t}{\mu}\right)\right).$$

Hence we obtain

$$\|v_{n+1}\|_{p+1}^{p-1}(T_1 - t_{n+1})^{\mu}$$

$$\leq \|v_0\|_{p+1}^{p-1} - \frac{C_1}{\mu + 1}(T_1 - \Delta t)^{\mu}\left(T_1 + \frac{\Delta t}{\mu}\right)$$

$$+ \frac{C_1}{\mu + 1}(T_1 - t_{n+2})^{\mu}\left(T_1 - t_{n+1} + \frac{\Delta t}{\mu}\right).$$

If v_0 satisfies

(4.19) $$\|v_0\|_{p+1}^{p-1} < \frac{C_1}{\mu + 1}(T_1 - \Delta t)^{\mu}\left(T_1 + \frac{\Delta t}{\mu}\right),$$

then the right side of this inequality will be null for $t_{n+1} = T'_{e_3}$ such that

$$\left(T_1 - T'_{e_3} + \frac{\Delta t}{\mu}\right)(T_1 - T'_{e_3} - \Delta t)^{\mu}$$

$$= (T_1 - \Delta t)^{\mu}\left(T_1 + \frac{\Delta t}{\mu}\right) - \frac{\mu + 1}{C_1}\|v_0\|_{p+1}^{p-1}T_1^{\mu}$$

$$= (T_1 - \Delta t)^{\mu}\left(T_1 + \frac{\Delta t}{\mu}\right) - T_1^{\mu+1} + (T_1 - T_{e_3})^{\mu+1}.$$

Since T'_{e_3} is bounded, we deduce easily that $(T_1 - T_{e_3})^{\mu+1} - (T_1 - T'_{e_3})^{\mu+1} = O(\Delta t)$, which gives $T_{e_3} - T'_{e_3} = O(\Delta t)$ if the quantity $(T_1 - T_{e_3})$ is greater than a positive real δ (independent of Δt). Then, as $(T_1 - T_{e_3})^{\mu+1} = T_1^{\mu+1}(1 - \frac{\mu+1}{C_1 T_1})$, the estimate holds if there exists $\delta_1 > 0$ such that $\frac{\mu+1}{C_1 T_1} \leq 1 - \delta_1$ or if v_0 satisfies (4.16).

Besides, as $(T_1 - \Delta t)^{\mu}(T_1 + \frac{\Delta t}{\mu}) \geq T_1^{\mu}(T_1 - \mu\Delta t)$, the inequality (4.19) holds if $\|v_0\|_{p+1}^{p-1} \leq \frac{C_1}{\mu+1}(T_1 - \mu\Delta t)$; namely

$$\alpha \frac{r - 1}{p - 1}\|v_0\|_{\infty}^{r-p}\|v_0\|_{p+1}^{p-1} \leq (1 - \alpha\frac{p - 1}{p}\Delta t\|v_0\|_{\infty}^{r-p})C(\Omega).$$

The lower bound T_{e_1} of the extinction time proceeds from Theorem 4.4.

Theorem 4.11. *If $r \geq 2p - 1$ and if v_0 satisfies (2.25), then the solution of the numerical scheme vanishes in a finite time T_e^* such that $T_{e_1} \leq T_e^* \leq T_{e_3}$.*

Proof. In this case, we have $\alpha_1 = 0$ and $\alpha_2 = \alpha$; then by multiplying the inequality (4.17) by $(T_1 - t_n)^{\mu-1}$, we obtain

$$\|v_{n+1}\|_{p+1}^{p-1}(T_1 - t_n - \mu\Delta t)(T_1 - t_n)^{\mu-1} \leq \|v_n\|_{p+1}^{p-1}(T_1 - t_n)^{\mu} - \Delta t C_1(T_1 - t_n)^{\mu}.$$

Since the inequality

$$(T_1 - t_n - \mu\Delta t)(T_1 - t_n)^{\mu-1} \geq (T_1 - t_{n+1})^{\mu},$$

holds, we get

$$\|v_{n+1}\|_{p+1}^{p-1}(T_1 - t_{n+1})^{\mu} \leq \|v_n\|_{p+1}^{p-1}(T_1 - t_n)^{\mu} - C_1\Delta t(T_1 - t_n)^{\mu},$$

which implies

$$(4.20) \qquad \|v_{n+1}\|_{p+1}^{p-1} \leq T_1^\mu \|v_0\|_{p+1}^{p-1} - C_1 \Delta t \sum_{j=0}^{n} (T_1 - t_j)^\mu.$$

But,

$$\Delta t \sum_{j=0}^{n} (T_1 - t_j)^\mu \geq \int_0^{t_{n+1}} (T_1 - s)^\mu ds = \frac{1}{\mu+1} \Big(T_1^{\mu+1} - (T_1 - t_{n+1})^{\mu+1} \Big).$$

From this estimate and (4.20), we obtain

$$\|v_n\|_{p+1}^{p-1} \leq \Big(\|v_0\|_{p+1}^{p-1} - \frac{C_1 T_1}{\mu+1} \Big) \frac{T_1^\mu}{(T_1 - t_n)^\mu} + \frac{C_1}{\mu+1}(T_1 - t_n).$$

If $\|v_0\|_{p+1}^{p-1} \leq \frac{C_1 T_1}{\mu+1}$, then the right side of this inequality becomes null at $t_n = T_{e_3}$. Thus the solution vanishes in a finite time $T_e^* \leq T_{e_3}$ and from Theorem 4.4 such that $T_{e_1} \leq T_e^*$.

In the particular case $r = 1$ we obtain the following results analogous to the theoretical case (see [14]).

- If $\alpha < \lambda_1$, the numerical solution vanishes after a time T_e^* such that

$$T_{e_1} \leq T_e^* \leq \frac{p}{p-1} \frac{\lambda_1}{\lambda_1 - \alpha} \frac{\|v_0\|_{p+1}^{p-1}}{C(\Omega)}.$$

- If $\alpha = \lambda_1$, the numerical solution converges in $L^{p+1}(\Omega)$ to $\theta \rho_1$ with

$$\theta \rho_1 \leq \Big(\int_\Omega v_0^p \rho_1 dx \Big)^{\frac{1}{p}}.$$

- If $\alpha > \lambda_1$, then $\lim_{n \longrightarrow +\infty} \|v_n\|_{p+1} = +\infty$.

In the particular case $r = p$ we have the following inequalities (see [11]):

$$\frac{p-1}{p} \Delta t G(v_{n+1}) \leq \Big(1 + \alpha \frac{p-1}{p} \Delta t \Big) \|v_n\|_{p+1}^{p-1} - \|v_{n+1}\|_{p+1}^{p-1} \leq \frac{p-1}{p} \Delta t G(v_n).$$

Since the sequence $(G(v_n))_{n \geq 0}$ is decreasing, we obtain

$$\|v_n\|_{p+1}^{p-1} \geq \Big(1 + \alpha \frac{p-1}{p} \Delta t \Big)^n \Big(\|v_0\|_{p+1}^{p-1} - \frac{G(v_0)}{\alpha} \Big) + \frac{G(v_0)}{\alpha}$$

and

$$\|v_n\|_{p+1}^{p-1} \leq \Big(1 + \alpha \frac{p-1}{p} \Delta t \Big)^n \Big(\|v_0\|_{p+1}^{p-1} - \frac{G(v_n)}{\alpha} \Big) + \frac{G(v_n)}{\alpha}.$$

Therefore, if v_0 satisfies $\alpha \|v_0\|_{p+1}^{p-1} < C(\Omega)$, there exists a time $T_e = N \Delta t$ such that the solution of the numerical scheme becomes null and we have

$$\Big(1 + \alpha \frac{p-1}{p} \Delta t \Big)^N \leq \frac{C(\Omega)}{C(\Omega) - \alpha \|v_0\|_{p+1}^{p-1}}.$$

- If v_0 satisfies $\alpha \|v_0\|_{p+1}^{p-1} > G(v_0)$, then $\lim_{n \longrightarrow +\infty} \|v_n\|_{p+1} = +\infty$.

5. CONVERGENCE OF THE NUMERICAL SCHEME

Let $T^*_{\Delta t}$ be the time existence of the numerical solution. We define a piecewise linear approximation $u_{\Delta t}$ of the weak solution u of (1.1) by

(5.1)
$$u_{\Delta t} = v_n^p + \frac{t - t_n}{\Delta t}(v_{n+1}^p - v_n^p) \qquad \forall t \in [t_n, t_{n+1}], \ t \leq T^*_{\Delta T},$$

and we denote $T^* = \inf\limits_{0 < \Delta t < \Delta t_0} T^*_{\Delta t}(T^* \geq T_1)$.

Let $T < T^*$. Here we prove that the solution of the numerical scheme converges to the weak solution of (1.1) on the interval $[0, T]$.

We need the following two lemmas.

Lemma 5.1. *If $v_0 \in H_0^1(\Omega) \cap C^\epsilon(\overline{\Omega})$, $(\epsilon \in]0, 1[)$, then there exists a positive constant C depending on Ω, α, p, r such that*

(5.2)
$$\sum_{n=0}^{N} \int_{\Omega} v_{n+1}(v_{n+1} - v_n)(v_{n+1}^{p-1} - v_n^{p-1})dx \leq \frac{p-1}{p}\Delta t(1 + C\Delta t)(J^*(v_0) - J^*(v_N)).$$

Proof. By multiplying (3.1) by $(v_{n+1} - v_n)$ and integrating on Ω we get

(5.3)
$$\frac{p}{(p-1)\Delta t}\int_{\Omega} v_{n+1}(v_{n+1} - v_n)(v_{n+1}^{p-1} - v_n^{p-1})dx$$
$$= \left[-\frac{1}{2}\int_{\Omega}|\nabla v_{n+1}|^2\, dx + \frac{1}{2}\int_{\Omega}|\nabla v_n|^2\, dx + \alpha\int_{\Omega}v_n^r(v_{n+1} - v_n)dx\right]$$
$$+ \left[-\frac{1}{2}\int_{\Omega}|\nabla(v_{n+1} - v_n)|^2\, dx\right.$$
$$+ \alpha_2\int_{\Omega}v_n^{r-p}(v_{n+1} - v_n)(v_{n+1}^p - v_n^p)dx$$
$$\left. + \alpha_1\int_{\Omega}v_n^{r-1}(v_{n+1} - v_n)^2 dx\right].$$

By using the Young inequality, we have

$$\alpha\int_{\Omega}v_n^r(v_{n+1} - v_n)dx \leq \frac{\alpha}{r+1}\left(\|v_{n+1}\|_{r+1}^{r+1} - \|v_n\|_{r+1}^{r+1}\right);$$

hence

(5.4)
$$-\frac{1}{2}\int_{\Omega}|\nabla v_{n+1}|^2 dx + \frac{1}{2}\int_{\Omega}|\nabla v_n|^2 dx + \alpha\int_{\Omega}v_n^r(v_{n+1} - v_n)dx \leq J^*(v_n) - J^*(v_{n+1}).$$

Besides, we have

(5.5)
$$-\frac{1}{2}\int_{\Omega}|\nabla(v_{n+1} - v_n)|^2 dx \leq -\frac{\lambda_1}{2}\int_{\Omega}(v_{n+1} - v_n)^2 dx.$$

If $r \leq p$, then $\alpha_1 = \alpha$ and $\alpha_2 = 0$ and the second part of the second member of (5.3) may be bounded by

(5.6)
$$\int_{\Omega}\left(\alpha v_n^{r-1} - \frac{\lambda_1}{2}\right)(v_{n+1} - v_n)^2 dx.$$

Let us denote $\Omega_+ = \{x \in \Omega / v_n^{r-1}(x) > \frac{\lambda_1}{2\alpha}\}$; so the quantity (5.6) may be bounded by

$$\int_{\Omega_+} \alpha v_n^{r-1}(v_{n+1} - v_n)^2 dx \leq \alpha \left(\frac{2\alpha}{\lambda_1}\right)^{\frac{p-r}{r-1}} \int_\Omega v_n^{p-1}(v_{n+1} - v_n)^2 dx,$$

and from Lemma 4.3, we obtain

$$(5.7) \qquad \int_\Omega \left[\alpha v_n^{r-1} - \frac{\lambda_1}{2}\right](v_{n+1} - v_n)^2 dx \leq C\Delta t(J^*(v_n) - J^*(v_{n+1}))$$

with $C = 2\alpha \frac{p+1}{p}(\frac{2\alpha}{\lambda_1})^{\frac{p-r}{r-1}}$.

If $r > p$, then we denote $M = \sup_{n\Delta t \leq T} \|v_n\|_\infty$, and the second part of the second member of (5.3) is bounded by

$$(5.8) \qquad \int_\Omega \left(\alpha_1 M^{r-p} v_n^{p-1} + \alpha_2 p M^{p-1} v_n^{r-p} - \frac{\lambda_1}{2}\right)(v_{n+1} - v_n)^2 dx.$$

If we denote $\Omega_+ = \{x \in \Omega / \alpha_2 p M^{p-1} v_n^{r-p} > \frac{\lambda_1}{2}\}$, then this quantity is bounded by

$$\int_{\Omega_+} \left(\alpha_1 M^{r-p} v_n^{p-1} + \alpha_2 p M^{p-1} v_n^{r-p}\right)(v_{n+1} - v_n)^2 dx.$$

If $r < 2p - 1$, then we have the estimate $v_n^{r-p} \leq (\frac{2\alpha_2 p M^{p-1}}{\lambda_1})^{\frac{2p-r-1}{r-p}} v_n^{p-1}$ on Ω_+, and if $r \geq 2p - 1$, then we have the estimate $v_n^{r-p} \leq M^{r-2p-1} v_n^{p-1}$. So in these two cases, we obtain

$$\int_\Omega \left(\alpha_1 M^{r-p} v_n^{r-p} + \alpha_2 p M^{p-1} v_n^{r-p} - \frac{\lambda_1}{2}\right)(v_{n+1} - v_n)^2 dx$$

$$\leq C_1 \int_\Omega v_n^{p-1}(v_{n+1} - v_n)^2 dx$$

with $C_1 = \text{Max}(\alpha_1 M^{r-p} + \alpha_2 p M^{p-1}(\frac{2\alpha_2 p M^{p-1}}{\lambda_1})^{\frac{2p-r-1}{r-p}}, \alpha_1 M^{r-p} + \alpha_2 p M^{r-p})$.

By using Lemma 4.3 we deduce that

$$(5.9)$$

$$\int_\Omega \left(\alpha_1 M^{r-p} v_n^{r-p} + \alpha_2 p M^{p-1} v_n^{r-p} - \frac{\lambda_1}{2}\right)(v_{n+1} - v_n)^2 dx$$

$$\leq 2\frac{p+1}{p} C_1 \Delta t(J^*(v_n) - J^*(v_{n+1})).$$

Then from the inequalities (5.3), (5.4), (5.7), (5.9), we obtain the lemma.

Lemma 5.2. *If $N\Delta t \leq T$, then there exists a positive constant $C(p)$ independent of the initial condition such that*

$$(5.10)$$

$$\sum_{n=0}^N \int_\Omega \left|v_{n+1}^p - v_n^p - \frac{p}{p-1}\, v_{n+1}\left(v_{n+1}^{p-1} - v_n^{p-1}\right)\right| dx$$

$$\leq C\Delta t^{\frac{p-1}{p+1}}\left(J^*(v_0) - J^*(v_N)\right)^{\frac{p}{p+1}}.$$

Proof. We recall a result shown in [10]: there a positive constant $C(p)$ such that

$$\sum_{n=0}^{N} \int_{\Omega} \left| v_{n+1}^{p} - v_{n}^{p} - \frac{p}{p-1} v_{n+1}(v_{n+1}^{p-1} - v_{n}^{p-1}) \right| dx$$

$$\leq C(p) N^{\frac{1}{p+1}} \left[\left(\sum_{n=0}^{N} \int_{\Omega} v_{n+1}(v_{n+1} - v_{n})(v_{n+1}^{p-1} - v_{n}^{p-1}) dx \right)^{\frac{p}{p+1}} + \left(\sum_{n=0}^{N} \int_{\Omega} v_{n}^{p-1}(v_{n+1} - v_{n})^{2} dx \right)^{\frac{p}{p+1}} \right].$$

With Lemmas 5.1 and 4.3, the right side of this inequality is bounded by

$$C(p) N^{\frac{1}{p+1}} \left[\left(\frac{p-1}{p} \Delta t(1 + C\Delta t)(J^{*}(v_{0}) - J^{*}(v_{N})) \right)^{\frac{p}{p+1}} + \left(2\frac{p+1}{p} \Delta t \left(J^{*}(v_{0}) - J^{*}(v_{N})\right) \right)^{\frac{p}{p+1}} \right]$$

so, for $N\Delta t \leq T$, by $C\Delta t^{\frac{p-1}{p+1}} (J^{*}(v_{0}) - J^{*}(v_{N}))^{\frac{p}{p+1}}$.

Theorem 5.3. *The function $u_{\Delta t}$ converges when Δt tends to 0, to the weak solution of problem* (1.1) *in $C(0, T; L^{1}(\Omega))$.*

Proof. The function $u_{\Delta t}$ is bounded in $C(0, T; L^{\infty}(\Omega))$ from Lemma 3.4 and in $C(0, T; H_{0}^{1}(\Omega))$. Besides, in the same manner as in [10], we obtain that $\frac{d}{dt} u_{\Delta t} \in L^{2}(0, T; L^{1}(\Omega))$. Hence, there exists a subsequence, again labeled $u_{\Delta t}$, such that $u_{\Delta t}$ converges to some function u in $C(0, T; L^{q}(\Omega))$ with $q < \frac{2d}{d-2}$ if $d > 2$, $q < \infty$ if $d = 2$ and in $C(0, T; \Omega)$ if $d = 1$ ([19]).

It remains to prove that u is a weak solution of problem (1.1). Let φ be a test function in $C^{2}(\Omega \times (0, T)) \cap C^{1}(\overline{\Omega} \times [0, T]); \varphi(x, t) = 0$ for $x \in \partial \Omega$.

Multiplying the equality (3.1) by φ and integrating on Ω, we get

$$\int_{\Omega} \frac{p}{p-1} v_{n+1} \left(v_{n+1}^{p-1} - v_{n}^{p-1} \right) \varphi dx$$

$$+ \Delta t \int_{\Omega} (v_{n+1}A\varphi - \alpha_{1}v_{n+1}v_{n}^{r-1}\varphi - \alpha_{2}v_{n+1}^{p}v_{n}^{r-1}\varphi) dx = 0;$$

hence, for $T = N\Delta t$ we get

(5.11)

$$\sum_{n=0}^{N-1} \frac{1}{\Delta t} \int_{t_{n}}^{t_{n+1}} \int_{\Omega} \frac{p}{p-1} v_{n+1}(v_{n+1}^{p-1} - v_{n}^{p-1})\varphi dx dt$$

$$+ \sum_{n=0}^{N-1} \int_{t_{n}}^{t_{n+1}} \int_{\Omega} (v_{n+1}A\varphi - \alpha_{1}v_{n+1}v_{n}^{r-1}\varphi - \alpha_{2}v_{n+1}^{p}v_{n}^{r-1}\varphi) dx dt = 0.$$

The first term of this equality may be written as

$$\sum_{n=0}^{N-1} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_{\Omega} \left(\frac{p}{p-1} v_{n+1}(v_{n+1}^{p-1} - v_n^{p-1}) - v_{n+1}^p + v_n^p \right) \varphi dx dt$$

$$+ \int_0^T \int_{\Omega} \frac{d}{dt} u_{\Delta t} \varphi dx dt.$$

From (5.10) the first part of this equality tends to zero when $\Delta t \longrightarrow 0$ and the second part tends to

(5.12)

$$-\int_0^T \int_{\Omega} u \frac{d\varphi}{dt} dx dt + \int_{\Omega} u(x,T)\varphi(x,T) dx - \int_{\Omega} u_0(x)\varphi(x,0) dx.$$

We prove in a classical manner ([14]) that the second term of (5.11) when $\Delta t \longrightarrow 0$ tends to

(5.13)

$$\int_0^T \int_{\Omega} \left(u^{\frac{1}{p}} A\varphi - \alpha u^{\frac{r}{p}} \varphi \right) dx dt.$$

So u satisfies the equation

$$-\int_0^T \int_{\Omega} u \frac{d\varphi}{dt} dx dt + \int_0^T \int_{\Omega} \left(u^{\frac{1}{p}} A\varphi - \alpha u^{\frac{r}{p}} \varphi \right) dx dt$$

$$= \int_{\Omega} u_0(x)\varphi(x,0) dx - \int_{\Omega} u(x,T)\varphi(x,T) dx$$

and is a weak solution of (1.1)

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